**Our heuristic:**

Let blank space be number 0. In any n-puzzle, we do following things:

P: If 0 is in its wrong place, then there must be a number, say k, that 0 is in k’s right place. Swap 0 and k, each swap is counted as 1 step.

Q: If 0 is in its right place, check if the puzzle is solved. If not solved, swap 0 with the first non-zero misplaces number. Each swap is counted as 1 step.

We continuously do operation P and Q until the puzzle is solved. h(n) is the number

of steps needed to solve a puzzle at state n.

**Proof of consistency:**

Definition: 1. A(n) represents a state with 0 in its right place and n misplaced number(s).

1. B(n) represents a state with 0 in its wrong place and totally n misplaced number(s)(**including 0**)

Now we want to proof: A(n), h(A(n)) ≤ c(A(n), A(n)’) + h(A(n)’)

B(n), h(B(n)) ≤ c(B(n), B(n)’) + h(B(n)’)

1. Proving A(n), h(A(n)) ≤ c(A(n), A(n)’) + h(A(n)’):

1.1 If n = 0, the puzzle is already solved. n = 1 is impossible, since 0 is in the right place,

there are at least 2 numbers in the wrong places if the puzzle is not solved.

1.2 For n = 2 and onwards, when we do operation Q on A(n), it will change A(n) to B(n+1).

Because the number swapped to 0’s position remains wrong and now 0 is also in a wrong position. So the LHS of the inequation can be replaced by h(B(n+1)) + 1.

1.3 In our heuristic, c(A(n), A(n)’) = 1, because any step is a swap which costs 1.

1.4 A(n)’ is 1 step further than A(n), which is doing operation Q on A(n). As discussed in 1.2,

A(n)’ = B(n+1).

1.5 So the RHS of the inequation becomes 1 + h(B(n+1))

1.6 The inequation becomes h(B(n+1)) + 1 ≤ 1 + h(B(n+1)), which is obviously correct.

1. Proving B(n), h(B(n)) ≤ c(B(n), B(n)’) + h(B(n)’):

2.1 n is at least 2 because 0 is in a wrong position which means there must be another number whose right position is mistaken by 0.

2.2 For n = 2, which means there are only 0 and another number that are in wrong places. Now

h(B(2)) = 1(only need to swap 0 with the other wrong number). As the same reason as 1.3, c(B(n), B(n)’) = 1. B(n)’ is already the goal state so h(B(n)’) = 0. So the inequation becomes 1 ≤ 1 + 0 which is correct.

2.3 For n = 3, suppose the 3 wrong numbers are 0, s, t, the only situation that could happen is: s mistakes 0’s right position, t mistakes s’s right position and 0 mistakes t’s right position. So obviously h(B(3)) = 2, c(B(n), B(n)’) = 1, h(B(3)’) = 1. The inquation becomes 2 ≤ 1 + 1 which is correct.

2.2 For n = 4 and onwards, according to 1.2, h(A(n)) = h(B(n+1)) + 1, so h(B(n)) = h(A(n-1)) - 1.

2.2 The LHS can be replaced by h(A(n-1)) - 1.

2.3 As the same reason as 1.3, c(B(n), B(n)’) = 1

2.4 There are two situation when we do operation P on B(n):

2.4.1 After doing operation P on B(n), 0 is at its right place:

2.4.1.1 Now B(n) becomes A(n-2) after operation P, because the number which mistake 0’s position is now in its right position and 0 is now in its right position.

2.4.1.2 So B(n)’ = A(n-2)

2.4.1.3 So the inequation becomes h(A(n-1)) - 1 ≤ 1 + h(A(n-2))

2.4.1.4 According to 1.2, the inequation can be written as h(B(n)) ≤ 2+h(B(n-1))

=> h(B(n)) - h(B(n-1)) ≤ 2

2.4.1.5 The problem becomes proving that given any state B(n), we can transfer it to B(n-1) within 2 steps.

2.4.1.6 If after doing P to B(n), 0 is still in a wrong place, then the state becomes B(n-1). We finish the transformation in 1 step.

2.4.1.7 If after doing P to B(n), 0 is in its right place, then we have A(n-2). Then do Q to A(n-2), we will get state B(n-1). Thus we finish the transformation in 2 steps.

2.4.1.8 Thus h(B(n)) - h(B(n-1)) ≤ 2.

2.4.1.9 Thus the original inequation is correct.

2.4.2 After doing operation P on B(n), 0 is still in a wrong place:

2.4.2.1 Proof by induction: since n=2 and n=3 all satisfy the inequation(see 2.2 and 2.3), suppose n=2, 3, 4,….,k satisfy the inequation.

2.4.2.2 When n = k + 1, we want to prove h(B(k+1)) ≤ c(B(k+1), B(k+1)’) + h(B(k+1)’)

2.4.2.3 After operation P, 0 is still in a wrong place which means B(k+1) becomes B(k). Because operation P will certainly correct 1 or 2 numbers, since 0 is not corrected, there is only 1 number that is corrected.

2.4.2.4 The LHS becomes 1 + h(B(k))

2.4.2.5 As the same reason as 1.3, c(B(n), B(n)’) = 1. B(k+1)’ = B(k) according to 2.4.2.3

So the RHS becomes 1 + h(B(k))

2.4.2.6 So the inequation becomes 1 + h(B(k)) ≤ 1 + h(B(k)) which is correct.

3. Thus by 1 and 2, Q.E.D.